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The Tabulation of Mathieu Functions

Introduction

The equation known as the Mathieu equation, which we write, following Ince,

$$(1) \qquad d^2y/dx^2 + (a - 2\theta \cos 2x)y = 0,$$

and in which a and θ are constants, arises in applied mathematics in two main groups of problems. The first group, of which Mathieu's problem of the transverse vibrations of a taut elliptic membrane is typical, consists of boundary problems relative to some partial differential equation, such as the wave equation, where the boundary is an ellipse—or possibly a hyperbola. Here a fundamental requirement is usually that y shall admit the period 2π in x. With a prescribed value of θ, this restricts a to a discrete set of characteristic numbers, each associated, in the concrete problem, with the frequency of a normal mode of vibration or some analogous quantity. In the complete solution of this class of problem, solutions of the associated equation

$$(2) \qquad d^2y/dx^2 - (a - 2\theta \cosh 2x)y = 0,$$

will usually also be needed.

A typical example of the second group of problems is the modulation of a radio carrier wave. Here one is concerned with either a steady state or an initial value problem for an ordinary differential equation, and the independent variable x will frequently denote time. In this case both a and θ are prescribed. It is known that the general solution may be written

$$(3) \qquad y = Ae^{\mu x}\phi(x) + Be^{-\mu x}\phi(-x),$$

where A and B are arbitrary constants, $\phi(x)$ is periodic in x admitting the period 2π, and μ is a quantity depending on a and θ termed the 'characteristic exponent'. Interest centres primarily upon whether μ is real or imaginary[1]—in the former case the solution is 'unstable', whereas in the latter it is 'stable', i.e. remains bounded for all (real) values of x.

For the periodic solutions, $\mu = 0$, so that the curves in the a–θ plane corresponding to the characteristic numbers dissect this plane into regions such that adjacent regions correspond to stable and unstable solutions respectively. For both groups of problems, then, knowledge of the characteristic numbers as functions of θ is a primary need.

The periodic solutions

The periodic solutions of (1) which admit the period 2π fall into four classes, according as their Fourier series involve respectively cosines or

[1] For real values of a and θ, μ may be chosen, with no loss of generality, to be either purely real, or purely imaginary.

sines of even or odd multiples of x. They are defined by Ince as follows:

$$ce_{2n}(x, \theta) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2rx, \tag{4.1}$$

$$se_{2n+1}(x, \theta) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r + 1)x, \tag{4.2}$$

$$ce_{2n+1}(x, \theta) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r + 1)x, \tag{4.3}$$

$$se_{2n+2}(x, \theta) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r + 2)x, \tag{4.4}$$

corresponding to which the characteristic numbers are denoted in order by

$$a_{2n}, \qquad b_{2n+1}, \qquad a_{2n+1}, \qquad b_{2n+2}.$$

In every case satisfaction of the differential equation necessitates recurrence formulae connecting three (in the first, two) successive coefficients. Elimination of the ratios of successive coefficients leads to an equation (involving a continued fraction) which expresses the relation between a and θ. The continued fraction forms the basis of the technique developed by Ince and Goldstein for the computation of the characteristic numbers, and this technique seems generally to have been used by later computers.

When the characteristic numbers are known, the ratios of the coefficients are calculable, but to make the coefficients definite, some normalisation rule must be chosen.

Tables

1. E. L. Ince, "Tables of the elliptic-cylinder functions," R. So. Edinb., *Proc.*, v. 52, 1932, p. 355–423.

2. E. L. Ince, "Zeros and turning points of the elliptic-cylinder functions," *ibid.*, 1932, p. 424–433.

1 is believed to be the only published table giving the numerical values of the functions themselves, and we therefore consider it first, before others chronologically earlier or later.

The tabular contents of no. 1 are

T. I, Characteristic numbers, p. 363. This gives $a_{0(1)5}$ and $b_{1(1)6}$ for $\theta = [0(1)10(2)20(4)40; 7D]$.

T. II–XIII, p. 364–375. These give Fourier coefficients for $ce_{0(1)5}(x, \theta)$ and $se_{1(1)6}(x, \theta)$, to 7D, for the same values of θ as T. I.

T. XIV–XXV, p. 376–423. These give the functions $ce_{0(1)5}(x, \theta)$ and $se_{1(1)6}(x, \theta)$ for $x = 0(1°)90°$ and $\theta = [1(1)10; 5D]$, along with second differences.

An introduction, p. 355–362, defines the notation and the functions, outlines the methods of computation, and describes the contents of the tables.

The normalisation rule adopted is that, if y is any of the functions tabulated, then

$$(5) \qquad \int_0^{2\pi} y^2 dx = \pi.$$

It follows that no Fourier coefficient exceeds unity in absolute value. The sign is chosen to secure that the values of the *ce* functions and the derivatives of the *se* functions shall be positive at $x = 0$. As a consequence of these rules it follows that, as $\theta \to 0$, $ce_0(x, \theta) \to 1/\sqrt{2}$, while for $m \geqslant 1$, $ce_m(x, \theta) \to \cos mx$ and $se_m(x, \theta) \to \sin mx$.

Values of the functions for values of x in quadrants other than the first can be obtained from the tables, since the properties of the functions as regards symmetry and antisymmetry about $x = 0$ and $x = \pm \frac{1}{2}\pi$ are those of the trigonometrical functions to which they tend as $\theta \to 0$. Values for the corresponding negative values of θ can also be obtained, since changing the sign of θ in equation (1) is equivalent to replacing x by $(\frac{1}{2}\pi - x)$. It is to be noted that changing the sign of θ also, in effect, interchanges a_m and b_m for odd (but not for even) values of m.

Interpolation of the functions in the x-direction is readily accomplished by means of the differences provided, but interpolation as regards θ would seem not to be feasible at this interval, neither in the functions nor in the characteristic numbers and Fourier coefficients.

T. 1 seems to have been completely overlooked in America; the present writer has encountered no reference to it in any American source.

The tabular contents of no. **2** are

T. I. Zeros, p. 431. This gives the zeros in degrees and decimals to 4D, of the functions $ce_{2(1)5}(x, \theta)$ and $se_{3(1)6}(x, \theta)$, in the open interval $0 < x < \frac{1}{2}\pi$, for $\theta = 0(1)10(2)20(4)40$.

T. II. Turning points, p. 432–3. This gives the turning points of the functions $ce_{1(1)5}(x, \theta)$ and $se_{2(1)6}(x, \theta)$, for the same range of x and values of θ as in Table I.

The numerical results exhibit the asymptotic approach of ce_m and se_{m+1} as θ increases, which had been anticipated on theoretical grounds. The introduction contains formulae which give approximations to the zeros and turning points for large, and also for small, values of θ.

Mention must also be made of short tables of characteristic numbers in earlier exploratory papers by Ince, not all of which are incorporated in no. **1**. The papers are

3. E. L. INCE, "Researches into the characteristic numbers of the Mathieu equation," R. So. Edinb., *Proc.*, v. 46, 1925, p. 20–29.

4.—(second paper), *ibid.*, v. 46, 1926, p. 316–322.

5.—(third paper), *ibid.*, v. 47, 1927, p. 294–301.

In these papers Ince used as the Mathieu equation

$$(6) \qquad d^2y/dx^2 + (a + 16q \cos 2x)y = 0,$$

so that the a_{2n+1} and b_{2n+1} of the later table no. **1** are interchanged. The sequence of characteristic numbers, in order of magnitude, is here

$a_0a_1b_1b_2a_2a_3b_3 \cdots$ as against $a_0b_1a_1b_2a_2b_3a_3 \cdots$ in no. **1**. It is to be inferred that Ince became convinced of the desirability of the changes of parameter from q to θ, and of sign, during the course of the computations for no. **1**.

In no. **3** Ince develops the continued fraction technique for the computation of the characteristic numbers, exhibits it by numerical examples, and gives a table of numerical results. The table contains $a_0\ a_1\ b_1\ b_2$ and a_2 (in the earlier notation) for q = [0(.1)1; 5D]. A diagram exhibiting the characteristic curves in the q–a plane is also given (p. 28).

In no. **4** Ince explores more deeply an asymptotic formula for the characteristic numbers, valid when q is large. In a table (p. 321) $a_{0(1)5}$ and $b_{1(1)6}$ (older notation) are given for q = [0(.5)5; 6D]. Since the continued fraction process is one of successive approximation, it is a great advantage to be able to calculate directly a good first approximation, and herein lies the great value of this asymptotic development. It appears, however, that this does not discriminate between a's (or b's) adjacent in the (older) sequence, and is really an expansion for some intermediate characteristic number.

In no. **5** this last point is further explored, and the fact that the asymptotic expansion relates to the characteristic numbers of half-odd-integral order is shown by numerical instances. The tables, (I–IV, p. 296) contain $a_0\ a_{\frac{1}{2}}\ a_1$, $b_1\ a_{1\frac{1}{2}}\ b_2$, $a_2\ a_{2\frac{1}{2}}\ a_3$, and $b_3\ a_{3\frac{1}{2}}\ b_4$, for q = 2, 4.5, 8, 12.5, 18 (i.e. $\sqrt{(32q)} = k$ = 8(4)24).

Great as was Ince's published contribution to the tabulation of the Mathieu functions, it represents a considerable curtailment of what he had planned, and indeed accomplished. He had worked out, to at least 12 decimals, all the characteristic numbers which he published in no. **1**, and also, to at least 10 decimals, $a_{0(1)4}$ and $b_{1(1)4}$ for the values 64, 100 and 144 of θ. His Fourier coefficients were originally calculated (using a normalisation rule different from that ultimately adopted) to 12 decimals, and there exists a manuscript table of the twelve functions as tabulated in no. **1**, to 7 decimals, with second differences. The manuscripts have been loaned to the present writer by L. J. C.

6. S. GOLDSTEIN, "Mathieu functions," Cambridge Phil. So., *Trans.*, v. 23, 1927, p. 303–336.

What seems to be the first systematic attempt of any magnitude to give numerical results for the periodic Mathieu functions was published on unpaged folding sheets at the end of this memoir.

Goldstein writes the Mathieu equation in the form

$$d^2y/dx^2 + (4\alpha - 16q\cos 2x)y = 0, \tag{7}$$

so that he writes 4α where Ince writes a (and $4\alpha_m$ instead of a_m and $4\beta_m$ instead of b_m) and also uses the older parameter q. The tabular material consists of:

T. I–V, which give for ce_0, se_1, ce_1, se_2 and ce_2 respectively the values of α_0, β_1, α_1, β_2, and α_2, to 5D, along with the Fourier coefficients, for q = [.1(.1)1(.2)4, 5(5)30, 40(20)100(50)200; 5D].

In the memoir Goldstein drew attention to the unsuitability of the older 'normalisation' rule whereby the coefficient of $\cos mx$ in ce_m, or of $\sin mx$ in se_m, was taken as unity, and proposed the rule (5) above, except that he chose 2π as the value of the integral for the function ce_0, which then tends to unity as q tends to zero. Thus Goldstein's Fourier coefficients agree with Ince's, except that for ce_0 they are $\sqrt{2}$ times as great, while the characteristic numbers tabulated by Ince are four times those tabulated by Goldstein. Goldstein gives a very much wider range of $q(\theta)$ than Ince, or indeed any other known table. The only values which overlap with Ince's are

Goldstein	$q = .5$	1	2	3	4	5
Ince	$\theta = 4$	8	16	24	32	40

Goldstein makes considerable use of the asymptotic expansion due to Ince, and in the memoir gives many more terms of it than Ince had calculated. The asymptotic expansion is further investigated in

7. S. Goldstein, 'On the asymptotic expansion of the characteristic numbers of the Mathieu equation," R. So. Edinb., *Proc.*, v. 49, 1929, p. 210–223.

Here, after lengthy and complicated analysis, the dominant term of the difference between a_n and a_{n+1} and between a_{n+1} and b_{n+1} is determined. A second term is determined empirically and in T. III, p. 222, the values of $b_1 - a_0$, $b_2 - a_1$, $b_3 - a_2$, and $b_4 - a_3$ are compared with the values given by the two terms, for $q = [.5(.5)5; 6D]$. T. IV, p. 222, shows that these results are good for small values of q by comparing $\beta_1 - \alpha_0$ and $\beta_2 - \alpha_1$ with the two terms of the asymptotic expansion, for $q = [.1(.1)1(.2)2.6; 5D]$.

8. H. P. Mulholland & S. Goldstein, "The characteristic numbers of the Mathieu equation with purely imaginary parameter," *Phil. Mag.*, s. 7, v. 8, 1929, p. 834–840.

In this paper the authors put $q = is$ in (7) and give the following Tables:

T. Ia, p. 839. α_0 and α_2 for $s = [.02(.02).18; 6D]$.

T. Ib, p. 839. α_0, β_1, α_3 and β_4 for $s = .2(.2)2$; and β_2 for $s = [.2(.2).8; 6D]$.

It was found that α_0 and α_2, which for small s are real and distinct, become equal when $s = .1836\cdots$, and for greater values of s (at least as far as $s = 2$) are conjugate complex quantities. Similar remarks apply to β_2 and β_4. These facts explain the arrangement of, and what at first sight appear to be gaps in, the tables.

9. K. Hidaka, "Tables for computing the Mathieu functions of odd order $se_1(x, \theta)$, $ce_1(x, \theta)$, $se_3(x, \theta)$, $\cdots$ $se_7(x, \theta)$, and $ce_7(x, \theta)$, and their derivatives," Imperial Marine Observatory, Kobe, Japan, *Memoirs*, v. 6, no. 2, 1936, p. 137–157.

The author uses (essentially) Ince's notation and method of normalisation. The Tables (p. 142–157) give $b_{1(2)7}$ and $a_{1(2)7}$ for $\theta = [0(.1)2.3; 7D]$, along with the Fourier coefficients of the functions, B_{2r+1} and A_{2r+1}, and also those of the derivatives, $-(2r+1)B_{2r+1}$ and $(2r+1)A_{2r+1}$, to 7D.

The restriction to functions of odd order is apparently dictated by the requirements of a tidal problem for which the tables were computed. Hidaka asserts that the interpolation for θ is possible. A few tests by the present writer would indicate that while second differences may be adequate for the Fourier coefficients, the greatest possible accuracy in the characteristic numbers requires the use of fourth differences.

10. J. A. Stratton, P. M. Morse, L. J. Chu, & R. A. Hutner, *Elliptic Cylinder and Spheroidal Wave Functions*, New York, Wiley, 1942, xii, 127 p.

This consists of a photographic reproduction of a paper by Chu & Stratton, with the same title, published in *J. Math. Phys.*, v. 20, 1941, p. 259–309, repaged as p. 1–51 in the book, to which have been added some numerical tables and an explanation of them, photographically reproduced from typescript. It was reviewed by H.B. in *MTAC*, p. 157–160.

As far as the elliptic cylinder functions (Mathieu functions) are concerned, the equation used is

$$d^2S/d\phi^2 + (b - c^2\cos^2\phi)S = 0, \tag{8}$$

so that the relations between the parameters used here and those of Ince are

$$\theta = c^2/4, \qquad c = 2\sqrt{\theta}, \tag{9.1}$$

$$a = b - c^2/2, \qquad b = a + 2\theta. \tag{9.2}$$

The periodic solutions of (8) are defined by

$$Se_l(c, \cos\phi) = \sum{}' D_n^l \cos n\phi, \tag{10.1}$$

$$So_l(c, \cos\phi) = \sum{}' F_n^l \sin n\phi, \tag{10.2}$$

where the prime indicates summation over positive integral values of n of the same parity as l. It will be seen that *Se* and *So* correspond to *ce* and *se* in Ince's notation. The normalisation rules adopted are that when $\phi = 0$, *Se* and the ϕ-derivative of *So* are each unity, i.e.

$$\sum{}' D_n^l = 1, \qquad \sum{}' nF_n^l = 1. \tag{11}$$

It follows that the relations between these Fourier coefficients, D and F, and those of Ince, A and B, are

$$A_n^{(l)} = D_n^l/\sqrt{\sum{}'(D_n^l)^2}, \qquad D_n^l = A_n^{(l)}/\sum{}' A_n^{(l)}, \tag{12.1}$$

$$B_n^{(l)} = F_n^l/\sqrt{\sum{}'(F_n^l)^2}, \qquad F_n^l = B_n^{(l)}/\sum{}' nB_n^{(l)}. \tag{12.2}$$

The values of b corresponding to the solutions *Se* and *So* are denoted by b and b' respectively, and the correspondence between these and the characteristic numbers of Ince will be clear from the above.

The Tables devoted to elliptic cylinder functions contain (p. 77–85) $b_{0(1)4}$ and $b'_{1(1)4}$ [5S], and the corresponding Fourier coefficients D or F [4 or 5D], for $c = 0(.2)4.4$, the values .1, .5(1)4.5 being intercalated. The greatest Fourier coefficient is given to 5S, and the others to the same number of decimals.

11. S. Lubkin & J. J. Stoker, "Stability of columns and strings under periodically varying forces," Quarterly Appl. Math., v. 1, 1943, p. 215–236.

Since the interest of these authors is in stability, characteristic numbers serve their purpose, and the tables at the end of the paper are restricted to these. The differential equation is taken in the form

$$d^2f/d\theta^2 + (\alpha + \beta \cos \theta)f = 0, \tag{13}$$

so that the relations between their parameters and those of Ince are

$$\alpha = a/4, \qquad a = 4\alpha$$
$$\beta = -\theta/2\ (= -4q), \qquad \theta = -2\beta.$$

Corresponding to the sequence

$$a_0,\ b_1,\ a_1,\ b_2,\ a_2,\ b_3,\ \cdots\cdots$$

of the characteristic numbers as used by Ince, Lubkin & Stoker employ the notation

$$\alpha(C_0),\ \alpha(C_1),\ \alpha(S_1),\ \alpha(S_2),\ \alpha(C_2),\ \alpha(C_3),\ \cdots$$

The Tables (p. 232–235) contain

$$\alpha(C_0) \cdots \alpha(S_2), \quad \text{for} \quad \beta = [0(.2)4(.4)10(1)16(2)20;\ 5D],$$
$$\alpha(C_2) \cdots \alpha(S_6), \quad \text{for} \quad \beta = [0(.2)4(.4)6(2)20;\ 5D].$$

These Tables reproduce—to 5 instead of 7 decimals,—12 out of Ince's 20 values, and for the same orders. Goldstein's results have been used for comparison, where the overlap occurs (but see 'Discrepancies', below).

It is known that there cannot co-exist, for the same values of a and θ, two independent solutions of (1) both admitting the period 2π. In other words, the second solution corresponding to $ce_m(x, \theta)$ or $se_m(x, \theta)$ is not periodic. This second solution is rarely of physical importance, but is mathematically interesting—and elusive. One attempt to define and provide a basis for calculating a second solution is known.

12. S. Goldstein, "The second solution of Mathieu's differential equation," Cambridge Phil. So., *Proc.*, v. 24, 1928, p. 223–230.

The table (p. 230) gives coefficients $b_{2r+1}^{(2m+1)}$ for $r = 0(1)8$ and $m = 0(1)8$ for the second solution corresponding to $ce_1(x, q)$ with $q = 1$ [8D], the definition being

$$in_1(x, q) = 2xce_1(x, q) + \sum b_{2r+1}^{(2m+1)} \sin (2r + 1)x. \tag{14}$$

In this, the coefficient $b_{2r+1}^{(2m+1)}$ is the contribution to the total coefficient of $\sin(2r + 1)x$ due to $A_{2m+1}^{(1)}$.

13. Philip M. Morse & Pearl J. Rubenstein, *Tables of Elliptic Cylinder Functions*. Calculated in 1940, 3 p. of formulae and 17 p. of tables. Hectographed from typescript.

The writer is indebted to R.C.A. for a photoprint of these tables, which has enabled the following account to be given. The tables form part of

the basis for the corresponding tables in no. **10**, which have in some respects superseded them.

In connection with the periodic solutions $Se_m(c, \cos\phi)$ and $So_m(c, \cos\phi)$, the tables give, in addition to the separation constants b_m and b'_m, the quantities N_m and N'_m defined by

$$N_m = \int_0^{2\pi} (Se_m)^2 d\phi = \pi \sum{}'' (D_n^m)^2, \tag{15.1}$$

$$N'_m = \int_0^{2\pi} (So_m)^2 d\phi = \pi \sum{}'' (F_n^m)^2, \tag{15.2}$$

where the double prime indicates summation over all positive integers and zero, of the same parity as m, together with the factor 2 for the terms in which $n = 0$.

These tables are notable, however, for the fact that they give some numerical information as regards solutions of the associated 'hyperbolic' equation. The differential equation, equivalent to (2), is taken in the form

$$d^2R/d\psi^2 + (c^2 \cosh^2\psi - b)R = 0. \tag{16}$$

Clearly one solution of (16), with $b = b_m$, is

$$Se_m(c, \cosh\psi) = \sum{}' D_n^m \cosh n\psi. \tag{17}$$

This solution converges slowly if ψ is not small. There is, however, an alternative solution of (16) in the form of a series of Bessel functions, which has convenient asymptotic properties, and it is remarkable that the coefficients in this expansion are multiples of D_n^m. In fact, if

$$Re_m^1(c, z) = \sqrt{(\pi/2)} \sum{}' i^{m-n} D_n^m J_n(cz), \tag{18}$$

then

$$Se_m(c, \cosh\psi) = \sqrt{(2\pi)}\lambda_m Re_m^1(c, \cosh\psi). \tag{19}$$

The satisfaction of the differential equation (16) by the series in (18)—with $z = \cosh\psi$—depends upon the differential equation and recurrence relations satisfied by $J_n(cz)$. These are shared by the functions of the second kind, $N_n(cz)$ in the notation of JAHNKE & EMDE, or $Y_n(cz)$ in the notation of WATSON and the BAASMTC. Consequently there is an independent second solution of (16)

$$Re_m^2(c, \cosh\psi) = \sqrt{(\pi/2)} \sum{}' i^{m-n} D_n^m N_n(c \cosh\psi). \tag{20}$$

With $\psi = 0$ in this, μ_m is defined by

$$Re_m^2(c, 1) = \sqrt{(2\pi)}\mu_m, \tag{21}$$

and also C_m and γ_m by

$$C_m^2 = 2\pi\mu_m^2 + 1/2\pi\lambda_m^2, \qquad \cot\gamma_m = 2\pi\lambda_m\mu_m. \tag{22}$$

Corresponding to So_m, with the separation constant b'_m, the functions Ro_m^1 and Ro_m^2, and the constants N'_m, λ'_m, μ'_m, C'_m, γ'_m, are similarly defined.

The first group of tables gives b_m, to 5S, N_m, to 5S, $\sqrt{(2\pi)}\lambda_m$, to 5S, $\sqrt{(2\pi)}\mu_m$, to 4S, C_m, to 4S, $\gamma_m(^{\circ}\,^1)$, and the Fourier coefficients D_n^m, to 4D, for $m = 0(1)4$, $c = 0(.1).2(.2)4.4$ and 4.5, and the corresponding values for the odd functions for $m = 1(1)3$ and the same values of c.

The second group of tables is easier to describe. It consists of values of $Se_{0(1)4}(c, \cos\phi)$ and $So_{1(1)4}(c, \cos\phi)$ for $c = 0(.5)4.5$ and $\phi = 0°(5°)90°$; [4D].

A quite casual examination has shown numerous discrepancies between these tables and those of no. **10**, of which a few are not small. The reliability is therefore not high, but they contain the results of considerable numerical exploration, provide results that might be adequate for some purposes, and could be very valuable in giving first approximations to anyone who wished to achieve greater accuracy.

14. W. G. BICKLEY, *Characteristic Numbers of the Mathieu Functions*. Unpublished manuscript.

The present writer has for some years gradually compiled a table of values of the characteristic numbers for the values of $k = \sqrt{\theta} = 0(1)10$. The calculations have been carried to sufficient figures to determine twelve decimals of the characteristic numbers. Results are complete as far as a_7 while some for b_8 have been obtained.

Some graphs showing the dependence of the Fourier coefficients $B_{2r+1}^{(2m+1)}$ upon q are given in

15. I. LOTZ, "Korrektur des Abwindes in Windkanälen mit kreisrunden oder elliptischen Querschnitten," *Luftfahrtforschung*, v. 12, 1935, p. 250–264.

Lotz follows Goldstein no. **6** and gives, in figs. 17–19, p. 259, graphs of the first four or five Fourier coefficients for se_1, se_3 and se_5, for the range of $0 < q < 20$.

Although no numerical tables are given (readers being referred to no. **6** and no. **8** above) reference must be made to

16. E. JAHNKE & F. EMDE, *Funktionentafeln mit Formeln und Kurven*, third ed., Berlin, 1938; or American reprint, New York, Dover Publications, 1944.

P. 283–295 give a synopsis of the formulae, following the notation of Goldstein no. **6**.

Fig. 166, p. 286, gives the $\alpha - q$ curves for the ranges $-13 < q < 13$, $-6 < \alpha < 14$, for the characteristic numbers from α_0 to β_7.

Figs. 167–177, p. 288–293, give various graphs of the five functions, $ce_0(x, q)$ to $se_2(x, q)$ while figs. 178–181, p. 294, 295, exhibit graphically the results of Mulholland and Goldstein no. **8**.

Non-periodic solutions of the Mathieu equation depend on the characteristic exponent—cf. equation (3) supra. Some values associated with this exponent, and also some graphs of non-periodic solutions, are given in

17. J. G. BRAINERD & C. N. WEYGANDT, "Solutions of Mathieu's equation—I," *Phil. Mag.*, s. 7, v. 30, 1940, p. 458–477.

The Mathieu equation is used in the form

$$d^2y/dt^2 + \epsilon(1 + k\cos t)y = 0. \tag{23}$$

Two fundamental solutions, $g(t)$, $h(t)$, are defined, such that $g(0) = 1$, $g'(0) = 0$; $h(0) = 0$, $h'(0) = 1$. It is shown that

$$g(2\pi) = h'(2\pi) = \cos 2\pi\mu = b. \tag{24}$$

Numerical values of $g(t)$ and $h(t)$ were determined for a range of values of ϵ and k by numerical integration of the differential equation, and hence b was determined. Sample graphs show $g(t)$, $h(t)$, and their derivatives, for $\epsilon = 9$, $k = .9$. In

T. I, p. 469, values of b are given for $k = .1(.1).6, .8(.1)1$ and $3\sqrt{\epsilon} = [0(1)6, 8, 9; 3D]$.

Notation

The conflicting notations used by the various writers are confusing. After considerable thought and discussion, the notation for the canonical form of the Mathieu equation to be used in a forthcoming account of the functions and their applications is

$$d^2y/dz^2 + (a - 2q\cos 2z)y = 0$$

with Ince's notation for the characteristic numbers and for the functions, and with Ince's normalisation rule. The reason for the introduction of the coefficient 16 by Mathieu has lost its force, and the mixture of Greek and Roman letters in (1) cannot easily be defended. Physically, the square root of the numerical value of q is significant, and this is also useful in defining solutions of the associated hyperbolic equations. It is proposed to use k such that $k^2 = |q|$.

Further details, and definitions of a set of functions satisfying the Mathieu and its associated equations, will be given elsewhere.

Discrepancies

As far as the writer is aware, no error has so far been reported in no. **1** or no. **2**. One error in no. **6** has been reported, by Goldstein himself (in no. **7**). The value of α_1 for $q = 2$, should be 2.33382 and not 2.33817.

In the overlapping cases ($c = 2, 4$, $\theta = 1, 4$) the separation constants given by no. **10** are in agreement with the characteristic numbers of Ince. The labour of comparing the Fourier coefficients (differently normalised) has not been undertaken.

The overlapping values in no. **11** have been compared with those of no. **6** and no. **7**. As regards no. **6** the following discrepancies occur:

Goldstein no. **6**		*Lubkin & Stoker* no. **11**	
$q = 0.3$	$\alpha_0 = -0.50534$	$\beta = 1.2$	$\alpha(C_0) = -0.50535$
$q = 0.4$	$\alpha_0 = -0.77897$	$\beta = 1.6$	$\alpha(C_0) = -0.77898$
$q = 0.2$	$\alpha_1 = 0.55406$	$\beta = 0.8$	$\alpha(S_1) = 0.55906.$

Since Lubkin & Stoker had compared their results with Goldstein's, one may presume that they have checked the end figures of the first two of these. Differences indicate that in the third of the above, Goldstein is correct.

Comparison of no. **1** with no. **11** yields the following:

Ince no. **1**		*Lubkin & Stoker* no. **11**	
$\theta = 40$	$a_2 = -20.20794\,08$	$\beta = 20$	$\alpha(C_2) = -5.05198$ ($= 20.20792 \div 4$)
$\theta = 2$	$b_3 = 9.14062\,77$	$\beta = 1$	$\alpha(C_3) = 2.28515$ ($= 9.14060 \div 4$)
$\theta = 4$	$a_3 = 10.67102\,71$	$\beta = 2$	$\alpha(S_3) = 2.66777$ ($= 10.67108 \div 4$)
$\theta = 24$	$b_4 = 13.55279\,65$	$\beta = 12$	$\alpha(S_4) = 3.38817$ ($= 13.55268 \div 4$)
$\theta = 12$	$a_4 = 22.97212\,75$	$\beta = 6$	$\alpha(C_4) = 5.74803$ (error)
$\theta = 32$	$b_5 = 26.10835\,26$	$\beta = 16$	$\alpha(C_5) = 6.52721$ ($= 26.10884 \div 4$)
$\theta = 40$	$b_5 = 22.33214\,85$	$\beta = 20$	$\alpha(C_5) = 5.58302$ ($= 22.33208 \div 4$)
$\theta = 40$	$a_5 = 41.34975\,44$	$\beta = 20$	$\alpha(S_5) = 10.33749$ ($= 41.34996 \div 4$)
$\theta = 40$	$b_6 = 41.43300\,52$	$\beta = 20$	$\alpha(S_6) = 10.35813$ ($= 41.43252 \div 4$)

In the above, differences indicate that Lubkin & Stoker's value of $\alpha(C_4)$ for $\beta = 6$ is in error, and should be 5.74303. In the other cases the discrepancies are too small for the available differences to discriminate with certainty, but in view of the fact that all Ince's values were worked out to 12 decimals by a process of successive approximation, and bear evidence of careful checking, they are prima facie the more reliable.

Comparison of Hidaka no. **9** with Ince no. **1** shows two end-figure discrepancies

Ince no. **1**	*Hidaka* no. **9**
$\theta = 2$, $B_3^{(5)} = 0.12413\,61$	$0.12413\,60$
$\theta = 1$, $a_5 = 25.02085\,43$	$25.02085\,44$.

In the case of the second of these, the present writer agrees with Ince, having obtained 25.02085 43454 5. . .

Comparison of Hidaka no. **9** with Goldstein no. **6** shows one discrepancy.

Goldstein no. **6**	*Hidaka* no. **9**
$q = 0.2$ $-B_3^{(1)} = 0.16171$	$\theta = 1.6$ $-B_3^{(1)} = 0.16171\,80$

Comparison of Hidaka no. **9** with Lubkin & Stoker no. **11** confirms that $\alpha(S_1)$ for $\beta = 0.8$ should be 0.55406. It yields also the following.

Hidaka no. **9**	*Lubkin & Stoker* no. **11**
$\theta = 1.2$ $b_3 = 9.06485\,47$	$\beta = 0.6$ $\alpha(C_3) = 2.26622$ ($= 9.06488 \div 4$)
$\theta = 2.0$ $b_3 = 9.14062\,77$	$\beta = 1.0$ $\alpha(C_3) = 2.28515$ ($= 9.14060 \div 4$)

Additional References

For collected accounts of ranges of the theory of solutions of the Mathieu equation, reference may be made (in addition to nos. **6**, **10**, and **16** above) to

M. J. O. Strutt, *Lamésche, Mathieusche, und verwandte Funktionen in Physik undTechnik*, (*Ergebnisse der Mathematik*, v. 1, part 3), Berlin, 1932.

E. T. Whittaker and G. N. Watson, *Modern Analysis*, Cambridge, University Press, fourth ed. 1927; Amer. reprint, 1943.

E. G. C. Poole, *Introduction to the Theory of Linear Differential Equations*, Oxford, University Press, 1936.

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RECENT MATHEMATICAL TABLES

195[A, D].—J. G. Beckerley, "The calculation of arg Γ ($ia + 1$)," ***Indian J. Physics***, v. 15, and Indian Assoc. for the Cultivation of Science, ***Proc.***, v. 24, 1941, p. 229–232. 16.5 × 23 cm.

Γ(z) for complex values of z has been tabulated to a rather limited extent. The tables of Walter Meissner, 1939, were reviewed in *MTAC*, p. 177; in his *Tables of the Higher*

Mathematical Functions, v. 1, Bloomington, Indiana, 1933, p. 269f., H. T. DAVIS tabulated $1/\Gamma(re^{i\theta})$, to 12D, for $r = -1(.1) + 1$, $\theta = 0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°$; in C. P. WELLS & R. D. SPENCE, *J. Math. Phys.*, M.I.T., v. 24, Feb. 1945, p. 61, there is a table of $\Gamma[(3 + ia)/4]$ giving the argument and modulus to 4D, $a = 1(1)5$; JAHNKE & EMDE's reference, p. 21 of the 1943 edition, to a certain GINZEL table of $(x + iy)!$, is without foundation in fact.

Beckerley gives a 4-place table of arg $\Gamma(ia + 1)$ for $a = [0(.1)2; 4D]$, with indications of the methods of calculation. This function occurs frequently in problems involving positive energy hydrogen functions.[1] The author expresses the hope that "this table will be of use to physicists who are engaged in numerical calculations involving the 'Coulomb phase factors' which occur in the continuous spectrum wave functions of hydrogenic atoms."

R. C. A.

[1] A. SOMMERFELD, *Wave-Mechanics*, London, 1930, p. 290f.; G. GAMOW, *Structure of Atomic Nuclei and Nuclear Transformations*, Oxford, 1937, chap. IX, equation 15, p. 163; E. C. KEMBLE, *Fundamental Principles of Quantum Mechanics*, New York and London, 1937, p. 177.

196[A, F].—I_1. J. KAVÁN, *Rozklad Všetkých Čísel Celých od 2 do* 256 000 *v Prvočinitele*, and on the second title-page: *Tabula Omnibus a* 2 *usque ad* 256 000 *Numeris Integris Omnes Divisores Primos Praebens* (Observatorium Publicum, Stará Ďala, Czechoslovakia), Prague, 1934, xi, 514 p. 28.3 × 29.4 cm. See RMT **71**, *Scripta Mathematica*, v. 4, 1936, p. 338. I_2. *Factor Tables giving the Complete Decomposition into Prime Factors of All Numbers up to* 256 000..., with prefaces, English by B. ŠTERNBERK, and Latin by K. PETR, and an introduction by ARTHUR BEER, London, Macmillan, 1937. xii, 514 p. 28.3 × 29.4 cm. 42 shillings. II. BAASMTC, *Mathematical Tables, volume V. Factor Table giving the Complete Decomposition of all Numbers less than* 100,000 *prepared independently by* J. PETERS, A. LODGE and E. TERNOUTH, E. GIFFORD . . ., London, 1935, xv, 291 p. 21.6 × 27.9 cm. Published at 20 shillings; out of print. For brevity these tables will be referred to in what follows as Table I and Table II.

Many mathematical problems arising out of the war effort are concerned with the development of new expansions for very special functions. These expansions, in most cases, have rational coefficients. To maintain accuracy in the combination and checking of these expansions it is often best to keep denominators in factored form and, to reduce the rational numbers to their lowest terms, it is convenient to know the factors of the numerators also. In short, a good computer need not be a number theorist to profit by using a fair-sized factor table at appropriate times.

A number of small factor tables will indeed be found in machinist's handbooks and small volumes of collected tables. These usually extend to about 10000 and often omit multiples of 2, 3, and 5, the very numbers most frequently met with in work of the sort described above. These embarrassing omissions are a practical necessity in the great factor tables of the first ten millions, which are intended after all for number theorists. The two tables under review are unique in giving all the prime factors of every number in their respective ranges and at the same time being sufficiently extensive to take care of the 5- or 6-figure work most frequently required. A comparison of the two tables may be made as follows. Table I is of course more than two and a half times as extensive as Table II. However, this does not mean that the former is twice as useful as the latter. Anyone who inspects a much used copy of a factor table cannot fail to notice the frayed and soiled edges of the early pages and the comparatively new appearance of its later pages.

The two tables are well arranged, though quite differently. Table I is arranged like an ordinary table of logarithms, consecutive numbers being in adjacent columns which are

headed 0, 1, $\cdots$, 9, with 50 lines to the page. The lines are numbered at both ends to facilitate entering the rather wide (23 cm.) table. Table II is arranged so that consecutive numbers occupy the same column and adjacent lines, an opening being devoted (in most cases) to 700 numbers, as compared with 1000 in Table I. The left page of the opening gives the factors of numbers whose last two digits are 00, 01, $\cdots$, 49. The average computer will no doubt find the arrangement of Table I more familiar and natural. When the factors of a long series of consecutive numbers are needed (as in some number theory problems) Table II is more convenient to use. In case the number to be factored is actually a prime both tables print the number in bold-face type, thus assuring the user that he has entered the table correctly. This is a decided improvement over earlier tables in which primes are represented by a dash.

The printing of Table I is decidedly inferior to that of Table II. There is much irregularity in type and inking. The exponents of the primes are unnecessarily small and in some cases (e.g. $N = 123725$) are of two sizes. The printing job in Table II is beautifully done, the primes being in old type and the exponents in new type.

Both tables are extremely reliable, especially Table II, which is based on three independent manuscripts and an intense program of proof reading and comparison with previous tables. The history of factor tables shows a preponderance of printer's errors over author's errors. Table I, which took 17 years to prepare, was verified in manuscript by actually multiplying together the decompositions given. Had this been done on the proof sheets, the more serious of the two following errors might have been avoided:

p. 32 $N = 15280$, *for* $2^4 \cdot 3 \cdot 191$, *read* $2^4 \cdot 5 \cdot 191$;
p. 39 argument left column, *for* 1800, *read* 1870.

These errors were discovered by J. C. P. MILLER.

For the sort of work encountered in practical problems the numbers to be factored seldom contain large prime factors. Such numbers constitute a quite small minority, so that a small factor table devoted exclusively to such numbers may nevertheless have a considerable upper limit. Such a table has been published by Cunningham.[1] It gives the factorization of all numbers up to 100000 having no prime factor in excess of 11 on nine small pages, 1196 entries in all.

D. H. L.

NOTE BY S. A. JOFFE: In the Czech introduction by KAVÁN there are two errors in quotations from CAYLEY, *Collected Math. Papers*, v. 9, p. 462–463. On p. V, box, factors of 391 should be $17 \cdot 23$, *not* $17 \cdot 33$; Cayley was correct and Kaván copied the factor 23 incorrectly. On p. VI, first box, the number corresponding to 297 should be 180, *not* 198; here Kaván copied Cayley's error, p. 463.

[1] A. J. C. CUNNINGHAM, *Quadratic and Linear Tables*, London, Hodgson, 1927, p. 162–170.

197[C, D].—*Natural and Logarithmic Haversines*, "arranged from Bowditch, American Practical Navigator, 1938 Edition, Table 34. By permission of the HYDROGRAPHIC OFFICE, United States Navy Department." New York, Macmillan, 1943, 38 p. 14.2 × 21.2 cm. Paper cover, 30 cents.

This publication is misleading in more than one respect. In the source indicated, in the above quoted footnote, the five-place table is of logarithmic and natural haversines, 0° to 120°, at interval 15″; 120° to 135° at interval 30″; 135° to 180° at interval 1′. The natural haversines are in black-face type throughout. What is "arranged" is simply to abridge this table to every minute of arc throughout, five degrees to the page, with constant uniformity in face of type. Hence this does not justify the title-page legend, *Natural and Logarithmic Haversines* by PAUL R. RIDER . . . and CHARLES A. HUTCHINSON, even though this table also occupies p. 187–222 of *Navigational Trigonometry*, published by these authors in 1943.

The other criticism is of an error made by the Hydrographic Office, rather than by these authors. Nathaniel Bowditch (1773–1838) never had anything to do with such tables as those under review. In 1844, Nathaniel's son, J. I. Bowditch (1806–1889) edited a volume

of tables taken from the *New American Practical Navigator* of his father. These were called *Bowditch's Useful Tables* (247 small p.); of this volume at least a score of editions had been published up to 1932. The editions after 1868 were published by a department of the Government, now called the Hydrographic Office, and after 1880 the word *New* was dropped from the earlier title *New American Practical Navigator*. Up to 1903 neither the *Useful Tables* ("Bowditch's" had been dropped from the title), nor the *American Practical Navigator*, contained haversine tables. But both of these works had such tables in their 1911 edition. Furthermore, the Hydrographic Office in that year published the volume, with the partial title-page, "*Useful Tables from the American Practical Navigator* by Nathaniel Bowditch" (427 large p.), even though no Bowditch had anything whatever to do with most of the contents of the volume, including the haversine tables.

The most elaborate published table of haversines is that of J. C. HANNYNGTON, *Haversines, Natural and Logarithmic, used in Computing Lunar Distances for the Nautical Almanac*, London 1876, 327 folio p. This is a 7-place table (except for the first 36° of the logarithms), 0 to 180°, log haversines at interval 15″, and natural haversines at interval 10″. For the range 0 to 125° or 135° a similar excellent table was published earlier, RICHARD FARLEY, *Natural Versed Sines from* 0 *to* 125°, and *Logarithmic Versed Sines from* 0 *to* 135° ⋯, London, 1856, 90 p. And yet earlier another 7-place table of haversines, at interval 10″, from 0 to 120°, JAMES ANDREW, *Astronomical and Nautical Tables with Precepts for finding the Latitude and Longitude of Places* ⋯, London, 1805, T. XIII, p. 29–148. There is a copy of Andrew's very rare work in the New York Public Library.

R. C. A.

198[C, E].—J. R. HULL & R. A. HULL, "Tables of thermodynamic functions of paramagnetic substances and harmonic oscillators," *J. Chem. Phys.*, v. 9, 1941, p. 465–469. 19.5 × 26 cm.

This paper, which was not mentioned in *MTAC*, p. 119, contains tables of the Einstein functions z/Z, $\ln Z$, z^2e^z/Z^2 where $Z = e^z - 1$. The tables give 5S for values of z between 0 and 14.4 at intervals ranging from .02 to .8. There is a misprint in the heading of the fifth column of p. 468. Tables are also given for Q and $\ln Q$, where

$$Q = \sum_{m=-j}^{j} e^{mx};$$

$j = \frac{1}{2}\,(1)\,2\frac{1}{2}$ is the angular momentum quantum number for the substances considered, and x ranges from 0 to 6. Actually x = [0(.02).2(.04).28(.02).32(.04).4(.1)4(.2)6; 5S].

H. B.

199[E].—P. B. WRIGHT, "Resistive attenuator, pad and network, theory and design," part 3 of a 4-part paper, *Communications*, v. 25, Jan. 1945, p. 57, 58, 60. 19.5 × 27.1 cm. Compare RMT **174**, p. 358.

Tables for $20 \log e^{\theta}(e^{2\theta} = k^2 > 1)$ = 0(.01).2(.05).4(.1)4(.5)30(1)60(5)140, 150, of (a) $\sinh 2\theta = (k^4 - 1)/2k^2$; (b) $\tanh^2\theta = [(k^2 - 1)/(k^2 + 1)]^2$; (c) $(1 - e^{-\theta}) = (k - 1)/k$; (d) $\sinh^2\theta = (k^2 - 1)^2/4k^2$; (e)[1] $\cosh 2\theta = (k^4 + 1)/2k^2$; (f) $\tanh^2 \frac{1}{2}\theta = [(k - 1)/(k + 1)]^2$; (g) $\operatorname{csch} 2\theta$; (h) $\coth^2\theta$; (i) $1/[1 - e^{-\theta}]$; (j) $\operatorname{csch}^2\theta$; (k) $\operatorname{sech} 2\theta$; (l) $\coth^2 \frac{1}{2}\theta$. These tables are to 5–9S. In part 1, of this 4-part paper, Aug. 1944, v. 24, no. 8, p. 52, 54, 56 there are tables for the same range of argument for (a) $\ln k$, $\frac{1}{2}\ln k$, $2\ln k$; (b) k, $1/k$, k^2, $1/k^2$; (c) $(k - 1)$; $2(k - 1)$, $(k - 1)^2$; (d) $1/(k - 1)$, $\frac{1}{2}/(k - 1)$.

[1] The right-hand member is given incorrectly on each of the three pages as $(k^2 + 1)/2k^2$.

200[E, M].—F. STÄBLEIN & R. SCHLÄFER, "Numerische Berechnung von $y(x) = e^{-x^2}\int_0^x e^{t^2}dt$," *Z. angew. Math.*, v. 23, Feb. 1943, p. 59–61. 21.5 × 27.7 cm.

Here is a table of $y = e^{-x^2}\int_0^x e^{t^2}dt$, for $x = [0(.1)10; 4D]$, which satisfies the differential equation $y' + 2xy = 1$. There are no references to other tabulations of this function, such as by TERAZAWA, and MILLER & GORDON, that of the latter being considerably more extensive than the one under review; see *MTAC*, p. 323.

201[L].—R. P. BALDWIN, "Tables of functions used in determinations of stellar ionization temperatures," Northwestern Univ., Dearborn Observatory, *Annals*, v. 4, part 14, 1940, p. [3].

Here is a table of $\int_{x_0}^{\infty} x^3dx/(e^x - 1)$, mostly to 4 or 5S, for $x_0 = 1(.5)12.5$. Compare *MTAC*, p. 140, **46**, and p. 189, RMT **152**.

202[L].—K. E. BISSHOPP, "Stress coefficients for rotating disks of conical profile," Am. So. Mech. Engrs., *J. Appl. Mech.*, v. 11, 1944, A-8—A-9. 21.6 × 26.6 cm.

In the calculation of stresses it is found to be advantageous to use both the fundamental solutions of the hypergeometric equation

$$x(1 - x)P'' + [c - (a + b + 1)x]P' - abP = 0,$$

$a + b = 1$, $ab = \sigma - 1$, $c = 3$ for the point $x = 0$; and also those for the point $x = 1$. In equation (15) the power series for $t^2F(a + 2, b + 2; 3; t)$ is given incorrectly but the calculations for the tables seem to have been made with the correct formula. The tables give the values of P_1, $-P_1'$, P_2, $-P_2'$, for $\sigma = 0.3$ and 0.36, $x = [0(.01)1; 6S]$ and it is thought that they are accurate to within 5 parts in 2,000,000. The functions P are defined as follows:

$$P_1(x) = S_1(x) = F(a, b; 3; x) = C_2S_2(t) + C_1\bar{S}_2(t),\ t = 1 - x;$$

$$P_2(x) = S_2(t) = t^2F(a + 2, b + 2; 3; t) = C_3S_1(x) + C_1\bar{S}_1(x);$$

$$\bar{S}_1(x) = -\tfrac{1}{2}a(a + 1)b(b + 1)S_1(x)\ln x + x^{-2} - (a + 1)(b + 1)x^{-1} - \sum_{n=2}^{\infty} (a - 2)_n(b - 2)_n x^{n-2}\Phi_n/[n!(n - 2)!];$$

$$\Phi_n = \Psi(a + n - 3) + \Psi(b + n - 3) - \Psi(n - 2) - \Psi(n)$$

$$\bar{S}_2(t) = 1 - abt - \sum_{n=2}^{\infty} a_nb_nt^n\Psi_n/[n!(n - 2)!] - \tfrac{1}{2}a(a + 1)b(b + 1)S_2(t)\ln t;$$

$$\Psi_n = \Psi(a + n - 1) + \Psi(b + n - 1) - \Psi(n - 2) - \Psi(n)$$

$$C_1 = (2/\pi)\,\mathrm{Sin}\,(a\pi)/[a(a + 1)b(b + 1)],$$

$$C_2 = (2/\pi)\,\mathrm{Sin}\,(a\pi)\left[\frac{1}{a} - \gamma - \Psi(a)\right] - \mathrm{Cos}\,(a\pi),$$

$$C_3 = \frac{\mathrm{Sin}\,a\pi}{\pi}\left[\frac{1}{a - 2} + \frac{1}{a - 1} + \frac{1}{a} - \frac{1}{a + 1} - 2\gamma - 2\Psi(a) - \pi\,\mathrm{Cot}\,(a\pi)\right]$$

γ = Euler's constant. $\Psi(x) = \Gamma'(x + 1)/\Gamma(x + 1)$.

Calculations were actually made for $\sigma = .24(.03).36$. Copies of the unpublished tables may be had on application to Fairbanks, Morse & Co., Beloit, Wisconsin.

The tables also include values of the stress coefficients p_1, p_2, p_3, q_1, q_2, q_3.

H. B.

203[L].—A. BLOCH, On the temperature coefficient of air-cored self-inductances," *Phil. Mag.*, s. 7, v. 35, Oct. 1944, p. 704. 16.9 × 25.3 cm.

Tables of $F_1 = (x^2/8)(1 - \Phi_2)$, $G_{1B} = (x^2/8)(1 - \Phi_1)$, $x = [0(.1)10; 5D]$, where $-\Phi_2 - i\Psi_2 = J_3(x\sqrt{-i})/J_1(x\sqrt{-i})$, $-\Phi_1 - i\Psi_1 = J_2(x\sqrt{-i})/J_0(x\sqrt{-i})$, or $\Phi_1 = 1 - 2W(x)/[xX(x)]$, $\Phi_2 = 1 - 4Z(x)/[xV(x)]$. X, V, W, Z, W/X, and Z/V have been tabulated by H. G. SAVIDGE, *Phil. Mag.*, s. 6, v. 19, 1910, p. 55f.; compare *MTAC*, p. 256.

204[L].—H. BUCHHOLZ, "Die Ausbreitung der Schallwellen in einem Horn von der Gestalt eines Rotationsparaboloides bei Anregung durch eine im Brennpunkt befindliche punktförmige Schallquelle," ***Annalen der Physik***, s. 5. v. 42, Mar. 1943, p. 433–435. 12.9 × 20.5 cm.

$$\partial m_{i\tau}^{(0)}(i\zeta)/\partial\zeta = m_{i\tau}'^{(0)}(i\zeta). \text{ For } \tau < 0,\ \zeta > 0, \text{ or for } \tau > 0,\ \zeta < 0,$$

$$m_{i\tau}'^{(0)}(i\zeta) = \tfrac{1}{2}i\,(\tfrac{1}{2}\pi)^{\frac{1}{2}}\cdot\left\{\frac{-4\tau J_1(2\sqrt{|\tau\zeta|})}{2\sqrt{|\tau\zeta|}} + \sum_{\lambda=0}^{\infty}\left(\frac{1}{4\tau}\right)^{\lambda}\cdot\gamma_\lambda^{(0)}(\zeta)(2\sqrt{|\tau\zeta|})^{\lambda}J_\lambda(2\sqrt{|\tau\zeta|})\right\};$$

$$\text{for } \tau > 0,\ \zeta > 0, \text{ or for } \tau < 0,\ \zeta < 0,$$

$$m_{i\tau}'^{(0)}(i\zeta) = \tfrac{1}{2}i\,(\tfrac{1}{2}\pi)^{\frac{1}{2}}\cdot\left\{\frac{-4\tau I_1(2\sqrt{(\tau\zeta)})}{2\sqrt{(\tau\zeta)}} + \sum_{\lambda=0}^{\infty}\left(-\frac{1}{4\tau}\right)^{\lambda}\cdot\gamma_\lambda^{(0)}(\zeta)(2\sqrt{(\tau\zeta)})^{\lambda}I_\lambda(2\sqrt{(\tau\zeta)}\right\},$$

where $\gamma_\lambda^{(0)}(\zeta)$ is a given function of ζ and λ.

Of the function $(2/\pi)^{\frac{1}{2}}m_{i\tau}'^{(0)}(i\zeta)$ there is a perspective graph $0 < \zeta < 6$, $-3 < \tau < +3$; and a table $\zeta = 1(1)6$, $\tau = [-3(.5) + 3; 5 \text{ or } 6S]$. Also there is a table of the first three zeros, τ_1', τ_2', τ_3' of $m_{i\tau}'^{(0)}(i\zeta)$, $\zeta = 0(1)6$; τ_1' to 5D; τ_2' to 4D, τ_3' to 2D. There are also graphs of the zeros $-4 < \tau' < +7$, $0 < \zeta < 6$.

205[L].—H. BUCHHOLZ, "Die Ausstrahlung einer Hohlleiterwelle aus einem kreisförmigen Hohlrohr mit angesetztem ebenen Schirm," ***Archiv. f. Elektrotechnik***, v. 37, March 1943, p. 160, 161, 163. 19 × 27 cm.

There is a table of $J_\nu(j_{0,s})$ for $s = 1, 2, 3$ and $\nu = [.5(.5)20.5; 7D]$; for $s = 1$ and $\nu > 10.5$, $s = 2$ and $\nu > 16$, the values of the function are all zero. The graphs of $J_\nu(j_{0,s})$, $s = 1, 2, 3$, are for $0 < \nu < 13$. On p. 163 is a table of the exact values of $D_s^{(0)}(p)$, for $s = 0(1)7$, $p = 0(1)7$, $s + p \not> 7$; and of $C_s^{(0)}(p)$, for $s = 0(1)6$, $p = 1\,(1)7$, $s + p$ successively $\not> 1, 3, 5, \cdots, 13$, where

$$D_s^{(0)}(p) = \sum_{\lambda=0}^{s}\frac{(2p+1)(2p+2)\cdots(2p+\lambda)}{(2p+2\lambda+1)\lambda!},$$

$$C_s^{(0)}(p) = \sum_{\lambda=0}^{s}\frac{(-1)^{\lambda}(p+1)(p+2)\cdots(p+\lambda)}{(2\lambda+1)(p-1)(p-2)\cdots(p-\lambda)}.$$

The expression for $D_s^{(m)}$ as given by Buchholz in (5.5b) is incorrect; *for* $(2p + 1)$, *read* $(2p + 1)_\lambda$.

R. C. A.

206[L].—J. COSSAR & A. ERDÉLYI, *Dictionary of Laplace Transforms.* Admiralty Computing Service, Department of Scientific Research and Experiment, London; Part 1, no. SRE/ACS.53, 1944, 42 leaves; Part 2A, no. SRE/ACS.68, Dec. 1944, 49 leaves; Part 2B, no. SRE/ACS.71, Feb. 1945, 56 leaves. 20.2 × 33 cm. Mimeographed on one side of each leaf. These publications are available only to certain Government agencies and activities.

Of this monograph, Part 1 contains an Introduction, and sections on: notes and abbreviations, general formulae, short table of Laplace transforms, bibliography. Only the

unilateral Laplace transformation is considered and the *Dictionary* consists of pairs of functions $f(t)$, $\phi(p)$, connected by the relation

$$\phi(p) = \int_0^\infty e^{-pt}f(t)dt = L\{f(t); p\}$$

called the Laplace transform of $f(t)$. In early days $\phi(p)$ was called the generating function of $f(t)$ and later writers spoke of Abel's generating functions. The extension of the term generating function from series to integrals was a natural development of the theory of generating functions of Lagrange and Laplace. The functions $f(t)$, $\phi(p)$ are classified according to $f(t)$ in Part 2, and according to $\phi(p)$ in Part 3 (which has not yet been published). In Part 2A $\phi(p)$ is given for each $f(t)$ of successive groups of functions: rational, algebraic, powers with arbitrary index, jump- and step-, exponential, logarithmic, trigonometric, hyperbolic, and composite elementary functions. In Part 2B, $f(t)$ includes the following forms: Bessel functions, modified Bessel functions, products of Bessel functions, Bessel integral functions and Fresnel integrals; Kelvin and Struve functions; sine, cosine, exponential and logarithmic integrals; Legendre, Gegenbauer, Jacobi, Hermite, and Laguerre polynomials; parabolic cylinder functions, Bateman k-function, and Whittaker functions; Legendre functions, Gauss's series, general hypergeometric series, functions of two or more variables; and theta functions. In many cases $\phi(p)$is expressed by means of the various types of Bessel functions, confluent hypergeometric functions, the incomplete gamma function and other functions of modern analysis.

No attempt is made at completeness. Not all available sources have been consulted; and even from the works consulted not all formulae have been included. Formulae which need much explanation, or which involve rare functions, have been omitted. Sometimes only a few typical examples are selected from an extensive list of related formulae. Then reference is made to the list where more involved formulae are found.

It is pointed out in the Bibliography that the most extensive published list of Laplace transforms in existence is in N. W. McLachlan & P. Humbert, *Formulaire pour le Calcul symbolique* (*Mémorial des Sci. Mathématiques*, v. 100), Paris, 1941, 67 p.

Each of these Dictionaries should be very useful.

H. B. & R. C. A.

207[L].—NYMTP, *Jacobi Elliptic Functions*, Washington, D. C. 1942 34 hektographed sheets and six mimeographed sheets, printed on one side only 21.5 × 35.5 cm. Not available for general distribution.

These tables were described *MTAC*, p. 125–126.

208[L].—NYMTP, Table in J. R. Whinnery & H. W. Jamieson, "Equivalent circuits for discontinuities in transmission lines," Inst. Radio Engrs., *Proc.*, v. 32, 1944, p. 114. 21.8 × 28 cm.

The table is of Hahn's function

$$S_0(a) = \sum_{n=1}^{\infty} \sin^2 (n\pi a)/(a^2n^3),$$

$a = [.01(.01)1; 4–5S]$. There are also graphs, $0 < a < 1$, of $S_0(a)$, and of the associated function

$$S_p(a) = \sum_{n=1}^{\infty} p^2 \sin^2 (n\pi a)/[n(n^2a^2 - p^2)], \; p = 1(1)8.$$

H. B.

209[L].—L. Schwarz, "Zur Theorie der Beugung einer ebenen Schallwelle an der Kugel," *Akustische Z.*, v. 8, 1943, p. 91–117.

The pressure on the surface of a rigid sphere under the influence of a plane wave of sound can be calculated by the known theory of Rayleigh. Calculations were made by the author with the aid of Miss E. Friedrichs of the real and imaginary parts of the functions ψ and

$$\psi e^{-i\omega} = -(2/\pi\omega)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(2n+1)i^{n+1}}{nH^{2}_{n+\frac{1}{2}}(\omega) - \omega H^{2}_{n+3/2}(\omega)} P_n(\cos\theta)e^{-i\omega},\ \omega = 2\pi a/\lambda = ka$$

for $\theta = 0(5^\circ)180^\circ$ and $\omega = 1(1)10$, 5D being given. Table I gives $|\psi|$ and T. II gives arc ψ for the same ranges. The first quantity represents the absolute value of the ratio $|p/p_0|$ on the spherical surface of the amplitudes of the diffracted and incident waves, the second quantity decreased by $\omega\cos\theta$ represents the difference in phase of these waves and is given in both radians and degrees. Tables are given also for the real functions $f_n(\omega)$, $g_n(\omega)$ defined by equation

$$H^{2}_{n+\frac{1}{2}}(\omega) = (2/\pi\omega)^{\frac{1}{2}} e^{-i\omega}\omega^{-n}[f_n(\omega) + ig_n(\omega)]$$

The ranges are $n = 0(1)15$ and $\omega = 1(1)10$. The tables of the quantities

$$A_n(\omega) = (2n+1)\frac{\omega^n}{\hat{f}_n^2 + \hat{g}_n^2}\cdot\hat{f}_n,\ B_n(\omega) = (2n+1)\frac{\omega^n}{\hat{f}_n^2 + \hat{g}_n^2}\cdot\hat{g}_n,$$

$$\hat{f}_n = nf_n(\omega) - f_{n+1}(\omega),\ \hat{g}_n = ng_n(\omega) - g_{n+1}(\omega),$$

are for $\omega = 1(1)10$ and for various values of n ranging from 0 to 22, 6D being given. A table with 5D is also given for $|p/p_0|^2$. Diagrams illustrate the numerical results. In fig. 1 relating to $|p/p_0|$ it is noted that the maximum at the south pole denotes 'optically speaking' that there is a bright spot on the face of the sphere away from the source. Plots of $|p/p_0| + \cdot 6\omega - 1$, $|p/p_0| + .04\theta^\circ$, $\phi^\circ - \cdot 4\theta^\circ$, $\phi^\circ - 2\theta^\circ$ against θ or ω are given to elucidate the properties of the functions tabulated.

H. B.

210[L].—H. STENZEL, "Über die Berechnung des Schallfeldes unmittelbar vor einer kreisförmigen Kolbenmembran," *Annalen d. Physik*, s. 5, v. 41, 1942, p. 256–259. 14 × 21.6 cm.

$$S_m(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_{m+\frac{1}{2}}(x),\ C_m(x) = (-1)^m(\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_{-m-\frac{1}{2}}(x);$$

T. 1 gives the values of

I. $\dfrac{1\cdot3\cdot5\cdot\ \ldots\ 2m-1}{2\cdot4\cdot6\cdot\ \ldots\ 2m}[S_{m-1}(10) + iC_{m-1}(10)]$, for $m = [1(1)12; 4D]$;

II. $(.4)^m\cdot J_m(4)$, for $m = [1(1)5; 4D]$;

I × II, for $m = [1(1)5; 4D]$;

III. $(.8)^m J_m(8)$, for $m = [1(1)12; 4D]$;

I × III, for $m = [1(1)12; 4D]$.

Table 2 gives the values of

I. $\dfrac{1\cdot3\cdot5\cdot\ \ldots\ 2m-1}{2\cdot4\cdot6\cdot\ \ldots\ 2m}\cdot J_{m+1}(10)$, for $m = [0(1)15; 4D]$;

II. $(10/12)^{m+1}[S_m(10) + iC_m(10)]$, for $m = [0(1)14; 4D]$;

I × II, for $m = [0(1)14; 4D]$;

III. $(10/16)^{m+1}[S_m(10) + iC_m(10)]$, for $m = [0(1)9; 4D]$;

I × III, for $m = [0(1)9; 4D]$. Compare STENZEL 1, *MTAC*, p. 233.

Table 3 gives the values of $p_a + ip_m$, to 3D, for $x = 10, 20, 40, 60$. $y = 0(1)20$. For $y < x$

$$p_a + ip_m = \sum_{m=1}^{\infty}\frac{1\cdot3\cdot5\cdot\ \ldots\ 2m-1}{2\cdot4\cdot6\cdot\ \ldots\ 2m}(y/x)^{m+1}J_{m+1}(y)[S_m(x) + iC_m(x)];\ \text{and for } y > x$$

$$p_a + ip_m = 1 - e^{-iy}J_0(x) - \sum_{m=1}^{\infty}\frac{1\cdot3\cdot5\cdot\ \ldots\ 2m-1}{2\cdot4\cdot6\cdot\ \ldots\ 2m}(x/y)^m J_m(x)[S_{m-1}(y) + iC_{m-1}(y)].$$

There are also graphs of p_a, p_m for $y = 10, 20, 40, 60$; $0 < x < 20$.

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in the introductory article of this issue, by Bickley (Goldstein, Hidaka, Lubkin & Stoker, Morse & Rubenstein), and in RMT **196** (Cayley, Kaván), **199** (Wright), **205** (Buchholz).

62. James Burgess, "On the definite integral $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$, with extended tables of values," R. So. Edinburgh, *Trans.*, v. 39, 1898, p. 257–321.

A. The great tables of this number of the *Transactions* do not begin until p. 283. In the earlier pages are various extended preparatory values of constants. Errors in these values in final digits are here summarized.

On p. 258 are 30-place values for $\frac{1}{2}\sqrt{\pi}$ and $2/\sqrt{\pi}$, and log $(2/\sqrt{\pi})$

$\frac{1}{2}\sqrt{\pi}$ *for* 670, *read* 671
$2/\sqrt{\pi}$ *for* 120, *read* 122.

These corrections are based on values of $\sqrt{\pi}$ and $1/\sqrt{\pi}$, computed to 317D and 310D, respectively (see *MTAC*, p. 200). Log $(2/\sqrt{\pi})$ is entirely free from error.

On p. 279 is a table of 31 constants and their logarithms. Of the 62 numbers comprising the table, each to 23D, only 12 were found to be entirely correct. Most of the errata are attributable to the fundamental error in Burgess' approximation to ρ. In order to correct this table each of the values was calculated to at least 32D.

	Constants		Logarithms	
	for	*read*	*for*	*read*
ρ	3 51	1 42	30 78	28 88
$1/\rho$	615 78	224 95	69 22	71 12
ρ^2	3 25	1 26	61 56	57 75
$\rho\sqrt{2}$	35 151 103 81	81 743 202 23	7 65	5 75
$2\rho\sqrt{\pi}$	009 806 981 30	298 912 773 92	20 15	18 25
$2\rho/\sqrt{\pi}$	9 82	7 46	8 88	6 98
$2\rho^2/\sqrt{\pi}$	59 638 137 627 19	00 537 595 982 03	459 350 094 499 66	359 350 094 495 86
$1/(\rho\sqrt{\pi})$	1 42	6 60	3 59	5 49
$1/(2\rho\sqrt{\pi})$	0 71	3 30	79 85	81 75
$\rho\sqrt{\pi}$	004 903 490 65	149 456 386 96	6 41	4 51

	for	*read*	*for*	*read*
$1/(\rho\sqrt{2})$	0 58	7 08	2 35	4 25
$\rho\pi^{1/6}$	3 01	0 48	5 99	4 09
$\rho(\pi/4)^{1/10}$	3 06	1 02	3 16	1 26
$\rho\sqrt{(\pi-2)}$	9 73	7 50	10 25	08 35
$\rho\sqrt{(15\pi-8)/6}$	4 65	2 47	4 00	2 10
$\rho\sqrt{(945\pi-128)/40}$	50 44	47 52	9 57	7 57
$\rho(4/3)^{\frac{1}{2}}$	5 56	3 15	7 00	5 10
$\rho(4/3)^{1/4}$	3 34	1 10	3 89	1 99
$\rho(113/45)^{\frac{1}{2}}$	6 05	2 74	40 23	38 33
$\rho(8/15)^{1/6}$	61 27	59 39	90 77	88 86
e^{ρ^2}	6 89	4 38	8 09	7 22
$e^{\rho^2}\sqrt{\pi}$	592 189 588 00	899 608 129 83	23 629 623 72	83 629 622 85
$e^{-\rho^2}$	2 06	3 65	1 91	2 78
$2e^{-\rho^2}/\sqrt{\pi}$	3 84	5 63	02	89
$e/2^{3/2}$	42	92		
$\sqrt{(\pi/2)}$	15 207 88	51 207 88		

The table on p. 281 was checked by comparison of Burgess' results with a 15-place table which I computed with the aid of NYMTP, *Tables of Probability Functions*, v. 1. The great

carelessness displayed in the preparation of the table is illustrated by log .08888591 given instead of log .088885991, and log. 272460716 instead of log .272462716.

t			$\log t$		
H	*for*	*read*	H	*for*	*read*
.1	85 991	55 990	.1	832 9230	686 7124
.3	6	5	.2	59	49
.4	49	59	.3	3 8936	7 0795
.6	79	81	.4	0986	1098
.7	9	8	.6	43	61
.9	3	4	.7	85	78
			.8	5	7
			.9	87	90

On p. 282 are given 30-place values of e^{-x}, and 27-place values (with one exception) of $2e^{-x}/\sqrt{\pi}$, for $x = 0(1)10$, and $\frac{1}{2}$. I recalculated each of these values to at least 38D. The value of e^{-x} for $x = \frac{1}{2}$ should end in 991 instead of 990. If for $x = \frac{1}{2}$ the value of $2e^{-x}/\sqrt{\pi}$ had been given to 27 instead of to 26 places, 367 should be substituted for 37. All other values in this table are correct. The value of $e^{-\frac{1}{2}}$ was computed to 80D and thus the value to 72D, given by Peters and Stein in the *Anhang* to Peters' *Zehnstellige Logarithmentafel*, v. 1, p. 12, was shown to be entirely correct.

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B.

page	t	error in	*for*	*read*
283	0.015	H	0.0179...	0.0169...
	0.017	$2e^{-t^2}/\sqrt{\pi}$	...53126	...53113
	0.055	H	...98209	...98333
284	0.110	argument	0.010	
	0.115	argument	0.015	
	0.155	$2e^{-t^2}/\sqrt{\pi}$	1.0815...	1.1015
	0.156	$2e^{-t^2}/\sqrt{\pi}$	1.0812...	1.1012
	0.157	$2e^{-t^2}/\sqrt{\pi}$	1.0809...	1.1009
	0.158	$2e^{-t^2}/\sqrt{\pi}$	1.0805...	1.1005
	0.159	$2e^{-t^2}/\sqrt{\pi}$	1.0802...	1.1002
	0.160	$2e^{-t^2}/\sqrt{\pi}$	...50273	...59273
	0.188	$2e^{-t^2}/\sqrt{\pi}$	...94388	...94288
285	0.291	H	...9220558	...9320558
286	0.367	H	...0689	...0679
287	0.429½	Δ_1	939383	938298
292	Heading	$2e^{-t^2}/\sqrt{\pi}$	$2e^{-t^2}/\sqrt{w}$	
	Heading	Δ_2	Δ	
	0.987½	Δ_1	...5541	...5549
296	1.011½	Δ_1	...615273...	...615373...
	1.036	$\log [2e^{-t^2}/\sqrt{\pi}] + 10$	...85280...	...85290...
298	1.122	$\log [2e^{-t^2}/\sqrt{\pi}] + 10$	...602099	...606210
299	1.154	$\log [2e^{-t^2}/\sqrt{\pi}] + 10$	...98049...	...98149...
301	1.250	$\log [2e^{-t^2}/\sqrt{\pi}] + 10$	...30829	...30833
302	1.308	H	...30256	...30286
	1.342	Δ_2	...36988	...36888
304	1.405	Δ_2	...406067	...406057
306	1.516	$\log [2e^{-t^2}/\sqrt{\pi}] + 10$	...8473	...8743
	1.564	H	...58239...	...58739...
308	1.798	H	...1899...	...1799...
310	1.966	H	...7457	...7447
	1.998	H	...226026	...326026
315	2.492	$\log [2e^{-t^2}/\sqrt{\pi}] + 10$	7.355468...	7.355458...
317	2.630½	Δ_1	...890673	...490673
318	2.782	Δ_4	91	1091
321	3.5	H	...8901628	...6901628
	4.1	H	...932999724	...993299972
	4.7	H	...980048	...970047

L. J. C.

C. *Integrand* $2e^{-t^2}/\pi^{\frac{1}{2}}$. For the following values of t the last figure should be (a) increased, (b) decreased, by a unit: (a) .187, 1.43, 3.3; (b) .947, 1.076, 1.077, 1.112, 1.230. P. 295, $t = 3$, *for* ...983, *read* ...947.

Integral H. For the following values of t the last figure should be (a) increased, (b) decreased by a unit: (a).397, 1.274, 1.276, 1.392, 2.504, 2.506, 2.510, 2.514, 2.552, 2.556, 2.628, 2.630, 2.634, 2.692; (b) .886, .927, .983, 1.260, 1.347, 1.466, 2.524, 2.642, 2.658, 2.662, 2.666, 2.668, 2.670, 2.898, 3.4.

P. 317,	$t = 2.644$, *for*	... 263, *read*	... 261,
	$t = 2.646$, *for*	... 857, *read*	... 855,
	$t = 2.648$, *for*	... 153, *read*	... 150,
	$t = 2.650$, *for*	... 978, *read*	... 976,
	$t = 2.652$, *for*	... 264, *read*	... 262,
	$t = 2.654$, *for*	... 056, *read*	... 054,
	$t = 2.656$, *for*	... 529, *read*	... 527,
	$t = 2.660$, *for*	... 963, *read*	... 961,
p. 321, H,	$t = 6$, *for*	...516 075, *read*	...519 737,
p. 321, G,	$t = 4.7$, *for*	... 544, *read*	... 545,
	$t = 6.0$, *for*	... 069, *read*	... 071.

On p. 314 the argument 2.880 should read 2.380.

NYMTP

EDITORIAL NOTE: In the above lists four errors in the 96 entries on p. 321 are noted (if two unit errors and one two-unit error, in the last figure are omitted). But a report of J. W. WRENCH JR. (to be published in our next issue) on the 24 entries of the L-column, shows that 13 are in error, several seriously. Accordingly J. O. IRWIN, (BAASMTC, *Mathematical Tables, volume VII, The Probability Integral*, by W. F. SHEPPARD, Cambridge, 1939, p. x), perhaps correctly states that this table is "seriously infested with error." In the value of H given by Burgess in his footnote on p. 321, *for* ...483 925, *read* ...480 263.

63. M. KRAITCHIK, *Recherches sur la Théorie des Nombres*, v. 1, Paris, 1924.

On p. 131–191 is Table I which gives the exponent γ of 2 mod p, for $p < 300,000$. For $p > 100,000$ I have discovered the following errata (p. 159–186):

p	γ	γ
104161	*for* 60	*read* 30
114601	2	6
121081	4	20
127681	8	152
267481	1	2

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64. E. C. J. v. LOMMEL, Bayer. Akad. d. Wissen., *math. natw. Abt., Abh.*, v. 15, 1886, p. 648, T.IIIa, Maxima and minima of the Fresnel integrals; also G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 1922 and 1944, p. 745. Compare MTE **58**, p. 366f.

The 32 values of this table have been completely checked and only the following four errata were found:

$S(2x/\pi)^{\frac{1}{2}}$			$C(2x/\pi)^{\frac{1}{2}}$		
x	*for*	*read*	x	*for*	*read*
6.283185($=2\pi$)	.343415	.343416	10.995574($=7\pi/2$)	.380389	.380391
15.707963($=5\pi$)	.600361	.600362	45.553093($=29\pi/2$)	.559088	.559087

J. W. WRENCH JR.

65. NYMTP, *Tables of Lagrangian Interpolation Coefficients*, New York, 1944. See *MTAC*, p. 314f.

P. 391, the entry corresponding to $n = 8$, $m = 1$, and $k = -2$, should be negative.

A. N. LOWAN

66. R. M. PAGE, 14000 *Gear Ratios* . . . , New York, The Industrial Press, 1942. See RMT **87**, p. 21f.

In MTE **53**, p. 326f, I gave a long list of the errors in this table found by Mr. S. JOHNSTON. We had hoped that the list would prove to be complete, but now Mr. F. LANCASTER, of Huddersfield, writes that he has checked Table 4, and found the following additional errors:

Page	N	*For*	*Read*
371	621	23 × 37	23 × 27
388	3904	59 × 66	Delete
391	4901	67 × 73	Delete
393	5432	46 × 118	Delete
400	9682	94 × 113	94 × 103

There are also three errors of position—less serious because they are unlikely to be misleading.

Page	N	
384	2860	52 × 55 should follow 44 × 65
398	8100	Transpose 81 × 100 and 75 × 108
401	10192	Transpose 98 × 104 and 91 × 112

L. J. C.

UNPUBLISHED MATHEMATICAL TABLES

Reference has been made to an unpublished table in RMT **202** (BISSHOPP); also to results by Ince and Bickley, *MTAC*, p. 412, 417.

34[A, B].—*Table of* $x^n/n!$, Manuscript prepared by, and in possession of, the NYMTP.

This table is for $x = 0(.05)5$, $n = 1(1)20$, to 10S.

A. N. LOWAN

MECHANICAL AIDS TO COMPUTATION

15[Z].—H. P. KUEHNI and H. A. PETERSON, "A new differential analyzer," *Electrical Engineering*, v. 63, May, 1944, p. 221–227. (Also in A.I.E.E., *Trans.*, v. 63, 1945, and discussion p. 429–431) 20.5 × 28.6 cm.

The article describes a differential analyzer of the Kelvin wheel-and-disc type which was built by the General Electric Company and put into service in Schenectady in 1943. The design follows closely that of the machine started in 1926 at Massachusetts Institute of Technology by Vannevar Bush, but incorporates a number of improvements which have been suggested by experience with later models, especially the one at the University of Pennsylvania. It has fourteen integrators, four manual input tables, and two double output tables; it can therefore be used for problems of considerable complexity. It is also arranged for operation as two independent units on simpler problems when not all of the elements are required.

The most important of the design innovations is the electronic arrangement used to relieve the integrator disc of mechanical load, and thus to minimize slipping of the integrator disc with respect to the wheel upon which it rolls. The arrangement uses two beams of light which pass through a polaroid disc mounted upon the integrator disc and through crossed

polaroid discs upon the output shaft which follows the integrator. Any angular difference in the positions of the integrator disc and output shaft causes one of the light beams to be attenuated more than the other. The difference is detected by balanced phototubes and used to control the speed of the motor which drives the output shaft. The entire torque of the motor is available at the output shaft, but the torque required from the rolling disc is only that necessary to overcome the friction of its jewelled bearings.

The electronic follower-system and other refinements have made it possible to operate the analyzer at higher speeds, thus cutting the time required for obtaining its graphical solutions.

PAUL L. MORTON

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NOTES

37. NOTATION yer$_n x$, yei$_n x$.—We have been informed by J. C. P. MILLER that ALAN FLETCHER, of the University of Liverpool, was the inventor of the notation

$$Y_n(xi^{3/2}) = \mathrm{yer}_n x + i\,\mathrm{yei}_n x,$$

which is used in the Liverpool *Index*. This note is in correction of the statements, *MTAC* p. 252, l. 23–24, and Corrigenda et Addenda p. 375.

38. A ROOT OF $y = e^y$.—In N **20**, p. 202f. (see also N **25**, p. 334f.) the solution was given of the transcendental equation 10 log $x = x$ or $x = 10^{x/10}$. In *Assurance Mag. and J.*, of the Institute of Actuaries, v. 3, 1853, p. 323, E. J. FARREN contributes an article "On the form of the number whose logarithm is equal to itself." The equation $y = e^y$ is considered, and it is found that

$$y = 1 + \frac{2}{2} + \frac{3\cdot3}{2\cdot3} + \frac{4\cdot4\cdot4}{2\cdot3\cdot4} + \frac{5\cdot5\cdot5\cdot5}{2\cdot3\cdot4\cdot5} + \ldots.$$

The derivation of this solution seems to have been due to ROBERT MURPHY.

R. C. A.

39. ROOTS OF $\tan x = xf(x)$.—Compare N **18**, p. 201f. In L. COHEN, "Alternating current cable telegraphy," Franklin Institue, *J.*, v. 195, 1923, p. 165f., there is a table, with 4–5S, of the first 8 roots of the equation

$$\tan x = 20x/(x^2 - 100).$$

H. B.

QUERIES

14. TABLES OF TAN^{-1} (m/n).—In some work I am now carrying on it is necessary to evaluate expressions of the form $\tan^{-1}(m/n)$, to 10D, where m and n are integers, ranging from 1 to 25 inclusive. For values of $n = 1, 2, 4, 5\ldots$, the values of the argument, and therefore the function may be found directly in NYMTP, *Tables of Arc Tan x*, Washington, D. C., 1942. Is there a table to cover the cases $n = 3, 7, 11\ldots$?

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QUERIES—REPLIES

15. Cube Roots (Q **11**, p. 372). In Vega-Hülsse, *Sammlung mathematischer Tafeln*, Leipzig, 1840, or 1849, or 1865, there is a table, p. 476–575, which has square roots and cube roots for $x = [1(1)10000$; square roots to 12D, cube roots to 7D]. The desired function in this Query may be obtained by multiplying the cube roots from 1000 to 2000 by $.04641\ 5888\ldots = 10000^{-1/3}$.

H. E. Salzer

NYMTP

Editorial Note: A cube-root table, of the same range as that of Vega-Hülsse, is given in editions of *Barlow's Tables* printed before 1930.

16. Rounding-off Notation (Q **10**, p. 335).—Devices to indicate something of the $n + 1$st place in an n-place table are desirable if the extra something is occasionally useful but generally to be ignored. For this purpose the high and low dots of Milne-Thomson & Comrie[1] (e.g., $2\frac{1}{2} \leq \mathbf{3.} \leq 2\frac{5}{6} \leq \mathbf{3} \leq 3\frac{1}{6} \leq \mathbf{3^{\cdot}} \leq 3\frac{1}{2} \leq \mathbf{4.} \leq 3\frac{5}{6}$, etc.) are to be preferred to either of the uses of the + sign referred to in Q **10**, since the former usage leaves the last figure of the n-place table as it should be (i.e., rounded off).

As an example of the possible utility of the M.-T. & C. device we may consider the applicability of the *American Air Almanac* to surface navigation; it is said to be used in preference to the *American Nautical Almanac* already 80% of the time, at least in the U. S. Navy. It is generally accepted that the error of astronomical sights is probably of the order of 5 to 15 minutes, or nautical miles, in the air, 1 or 2 only at sea. In order not to increase these errors the ephemerides and correction tables are given to the nearest minute in the *Air Almanac*, to the nearest tenth of a minute in the *Nautical Almanac*. It is clear in the first place that the former will satisfy the normal demands of sea navigation, and secondly that the greatest accuracy obtainable would be satisfied by about a third of a minute, rather than a tenth, in the tables. Accordingly, the introduction of the high and low dots into the *Air Almanac*,[2] together with the improvement of some of its correction tables,[3] would give it the accuracy needed in the most refined sea navigation without affecting its convenience for air navigation.

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[1] L. M. Milne-Thomson & L. J. Comrie, *Standard Four-Figure Mathematical Tables*, London, Macmillan, 1931.

[2] W. J. Eckert, in "Air Almanacs," *Sky and Telescope*, v. 4, p. 12–15, 17, Nov. 1944, shows that the French air almanac uses such a device, but one which, in the writer's opinion, is inferior to that of Milne-Thomson & Comrie, because the last figure must be altered in some cases, when accuracy to the nearest minute only is desired.

[3] The writer has discussed a new kind of "critical graph," which is used in the same manner as a critical table but can be read to greater accuracy, if need be, in "The *Air Almanac* refraction tables," U. S. Naval Institute, *Proc.* v. 70, Sept. 1944, p. 1140–1141; Univ. Calif., Los Angeles, *Astronomical Papers*, no. 5.

CORRIGENDA ET ADDENDUM

P. 305, J. Steiner 1_2, l. 1, *for* v. 1, *read* v. 11.

P. 391, l. −16, −15, *for* A = Airey, C = Comrie, M = Miller, *read* A = Airey, C = Comrie, M = Miller.

P. 392, 1909, 28 Line 9, *for* π^2, and π, *read* Π^2, and Π.

P. 394, 1909, 138, l. 2, move *A.M.*, 18 up to l. 1.

P. 397, l. −9 *for* γ, *read* C, and add: We here use $C = \ln \gamma$ for Euler's constant, in place of the γ commonly used by British writers.

P. 403, l. 3, *for* H. Böckh, *read* R. Böckh.

CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

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B. Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
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Professor Ketchum, Professor Lowan

F. Theory of Numbers
Professor Lehmer

G. Higher Algebra
Professor Lehmer

H. Numerical Solution of Equations
J. Summation of Series

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T. Chemistry
U. Navigation

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The addresses of all contributors to each issue of *MTAC* are given in that issue, those of the Committee being on cover 2. The use of initials only indicates a member of the Executive Committee.

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